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THE UNIFICATION OF FRESHMAN MATHEMATICS.

By JOSEF A. NYBERG.[1]

In a recent article,[2] Professor F. L. Griffin has presented a method for introducing calculus into the freshman year. A course as there outlined would, perhaps, be successful only in a college which is able to command an exceptionally good student body, and might possibly be a failure in a state university where the preparation of the entering classes is decidedly varied and poorer. In these schools, which must handle students of widely different preparations and abilities, it is not of so much importance to teach the calculus to the extent outlined by Mr. Griffin as to correlate the freshman work and to introduce enough differential calculus to be of service in the study of physics during the first months of the sophomore year. In this article I wish to outline a possible method for such schools. In the discussion following the outline, I consider its essential elements.

Part I is the usual introduction explaining such terms as scales, graphs, functions, ...

Part II is a study of the line and the elementary elliptic, parabolic, and hyperbolic curves, together with theorems on the symmetry and the translation of curves so that the general equation of the second degree without the xy term can be discussed.

Part III, divided into two sections, is first a study of trigonometry under five heads:

1. Definitions and fundamental relations between the six functions;
2. The radian measure of angles and polar coördinates;
3. The addition formulæ;
4. The relations between the six elements of a triangle;
5. The inverse trigonometric functions;

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[2] *An Experiment in Correlating Freshman Mathematics*, THE AMERICAN MATHEMATICAL MONTHLY, December, 1915.

and, secondly, a study of the applications:

1. The straight line (angles between lines, polar and normal forms of the equations, distance from a point to a line, . . .);
2. The solution of cubics with real roots;
3. Kinematics (uniform circular motion, simple harmonic motion, progressive waves).

Part IV is a presentation of logarithms:

1. The explanation of the exponential function a^x, and the significance of the constant e;
2. The use of logarithms in calculations (equations from scientific applications, compound interest, oblique triangles);
3. Logarithmic coördinates;
4. Hyperbolic functions and the catenary.

Part V is the introduction to differential calculus preceded by parts of college algebra:

1. Illustrations and review of periodic, single and multiple valued, odd and even, continuous and discontinuous functions;
2. Zeros of functions and simultaneous equations;
3. The remainder theorem and synthetic division;
4. The rate of increase of functions (derivatives of x^n, $\sin x$, $\cos x$, a^x);
5. The technique of differentiation with the usual elementary applications.

Part VI is the customary work on conics and other plane curves.

Part VII is vectors in their relation to forces and complex numbers.

The basis of this arrangement has been not so much the attempt to *correlate* the different subjects, as to take advantage of the *unity* which is present in mathematics and which we have hitherto overlooked. The best correlation will mean a fluctuation from one subject to another whenever two problems in distinct fields can be solved by similar methods or when a fundamental concept underlies different subjects, or when a result from one subject can be used to extend our knowledge of another. In this sense the above outline is itself a correlation, for some problems of lines are solved after the introduction of trigonometry, and graphs can be used in deriving trigonometric relations, etc. Unification, on the other hand, means the presentation of the different fields as parts of a single whole so that no part can be omitted without disturbing the completeness of the structure. The unity of the above outline will be more evident if I give the following titles to the divisions:

I. The aims and methods of mathematics.
II. The simple algebraic functions.
III. The trigonometric functions.
IV. The exponential and logarithmic functions.
V. General properties of functions (periodicity, zeros, derivatives).
VI. Irrational algebraic functions.
VII. Functions of two variables.

By an elaboration of these titles I shall try to show how the course is unified by keeping constantly in mind that mathematics is a study of relations between variables, or the properties of functions. Each division will then involve a study of a particular class of functions and we may jump from one subject to another only when the discovery of some relation enables us to extend our knowledge of a previous subject.

The first division, being the introduction, brings in the notion of variable and dependence of one variable upon another, shows how numbers arranged on a line (a scale) enable us to exhibit the correspondence between different values. The graph obtained by placing the scales at right angles should appear to the student only as an easy way of exhibiting this correspondence; and the teacher should emphasize the relation between the points on the scales rather than the points on the graph. The language of coördinates and axes is introduced apparently only as an excuse to simplify the terminology. By emphasizing the fact that a curve is an exhibition of the restriction on, or the relation between, the values of the variables, we can get away from the graphic algebra which teaches the plotting of curves and mistakes the means for the end.

The ideas of such an introduction comprise the first chapter of any text on coördinate geometry, and are presented sooner or later in every course on college algebra, and in all the recent texts on trigonometry. But in none of these places can the emphasis be properly placed on the functional relation. Geometry places the emphasis on coördinates and curves; trigonometry introduces the ideas merely to graph the functions and then stops; the algebras use the curves as geometric means of solving equations. Irrespective of what course the student is about to pursue there is no reason why the year should not begin with a thorough explanation of the notion of function.

The second division then begins with the simplest function, the linear. By presenting the line in its slope form we can introduce the notion of the rate of increase of a function and illustrate how a function is determined by its rate of increase and its value at one point. While this idea will not recur until much later there are nevertheless problems to which it can be applied, for example, the coefficient of linear expansion. We consider briefly the functions of the type x^n, and then formulate the general methods of curve tracing, for example, the change in a curve when in its equation x is replaced by kx or by $x - a$. These methods enable us to study the parabola, ellipse, and hyperbola (without the xy term) and incidentally can furnish a review of quadratics and exponents.

The third division considers what I prefer to call the trigonometric functions rather than trigonometry, for the latter term might not, for example, include polar coördinates except as a side-issue, whereas as in studying functions we would assuredly draw their graphs as an aid in finding their properties, and the polar graph is as useful as the cartesian. While in all teaching theory and application are blended, they are separated in the outline to conform with the idea that we seek the properties of functions first and then use these properties to enlarge our ideas as to previous problems. The applications carry us from field to field but we are thoroughly conscious of our travels and have a legitimate excuse for the journey. It is this consciousness of change and the deliberateness of it which will remove the difficulties pointed out by E. B. Wilson:[1] ". . . most problems . . . are really problems in algebra, in trigonometry, or in analytics, and the training in classification according to these topics is highly valuable;" for we do classify the problem and then focus on it the nature of the functions involved.

The treatment of the fourth division is along similar lines. We are not interested in logarithms alone, but in the logarithmic function. The subject can be well introduced by a problem involving a function whose rate of increase is proportional to itself, for example, that of determining the number of bacteria in a solution. Any approximate solution leads to an exponential function; the correct solution leads to a splendid explanation of the significance of the constant e and unquestionably the very best explanation a student can have. This has always been one of the weak spots in calculus; merely calling e the natural base does not reveal its significance. Logarithms may then be introduced as answering the question of the time required for the bacteria to reach a specified number. After the customary formal treatment of the uses of logarithms we consider the various exponential relations and the use of logarithmic coördinates. The exponential curve, the catenary, and the hyperbolic functions close the division.

To this point we have consistently presented the fundamental functions, x^n, the trigonometric, and the exponential. The fifth division shows again the unity of the subjects by treating the properties common to all the functions. At this stage the student has worked with a sufficient number of functions to profit by classifying them as odd, periodic, continuous, etc. This is also a suitable place for parts of algebra such as the zeros of functions, sets of zeros of two functions, the remainder theorem, and synthetic division. These subjects now enter as a natural step in finding the properties of functions.

The next part needs perhaps more careful attention than the others because we must make sure that the calculus is not introduced at the expense of other subjects and that the work will not need repetition during the following year. That no subject is slighted will be evident before the end of the article. Any method of correlation will, in fact, enable us to treat more subjects during the year because we have at our disposal the time gained through the avoidance of such repetition as is always unavoidable when algebra, trigonometry and analytics are taught in three distinct courses. Each of these courses, for example, must separately introduce the notions of functions and of coördinates; under the present outline this is done thoroughly once. Again, many of the results of analytics can be derived by the shorter methods of calculus, and this in itself is desirable unless we agree to prefer old methods to new ones merely because they are old.

If less than fifteen or even twenty hours are devoted to the calculus we may feel sure that the student will forget all about it before the following year.

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[1] A review of C. S. Slichter's *Elementary Mathematical Analysis* in the MONTHLY for February, 1915, page 56.

For a treatment in twenty-five hours the writer suggests a complete development of the technique of differentiation together with such applications as are found in curve tracing (maxima, minima, and inflexion points), maxima and minima of functions, and problems dealing with velocity and rates. No reference should be made to integral calculus except of course the customary exercises in finding $f(x)$ when its derivative is given and even these should be used only to impress on the memory the standard formulæ of differentiation. With this accomplished the sophomore will have a definite background and the time will not be wasted.

In the course outlined by Mr. Griffin the analytic geometry is postponed until the junior year. Unless, however, the student follows mathematics for three years, analytics can be properly used to fill in the remainder of the freshman year. It will be noticed in the present outline that the line, polar coördinates, and the simple conics have been treated before the sixth division is reached. The conics are now defined by their focal definitions ($r_1 + r_2 = 2a$, etc.), the existence of directrices proved, the eccentricities defined and compared. Then the latus rectum is introduced preliminary to the polar equation, $\rho(1 + e \cos \theta) = l$, which holds for all the conics. The change in the equation of a curve under rotations and the analysis of the general equation of the second degree follows. A study of equations involving arbitrary constants leads to the discussion of lines tangent to a conic where, if the teacher so chooses, the methods of calculus may be used instead of handling the discriminant of the quadratic. By using both methods or by treating these same problems in the previous division the student will see how the new theory improves old methods. The discussion of locus problems leads to such other curves as the witch, cissoid, cycloids, conchoids, etc., and the position of these subjects here is advantageous in that the end of the year will be approaching and so these subjects can be treated as briefly or as thoroughly as time permits.

The student in an engineering school, as a preparation for the sophomore year, needs a knowledge of vectors and this subject ends the year's work. As the theory involves always the use of the trigonometric functions, vectors might be studied as one of the applications of these functions. But the present arrangement preserves the unity of the course, inasmuch as the last division then deals with variables whose values can not be represented by points on a line. The position of vectors and complex numbers at the end of the year again offers possibilities for variation in treatment.

The various divisions of this outline constitute the essential parts of "*An Introduction to the Elementary Functions*," a fit title for this course. The fact that such a title is possible and appropriate is a last proof of the unity of the material. Of course a mere title is of slight value in teaching and any outline can fail unless the presentation has coherence, a thing which must be left to the textbook writer. Also, unless the teacher consistently adheres to the notion that every chapter is a study of the relations between variables, properties of functions and the applications of these properties, all unity is lost.

There are three types of students for whom a course as here outlined is especially suitable. The non-specialist who wishes an introduction to mathematics can profit less by trigonometry or algebra than by an immediate insight into the theory of functions. Unity is to him of more importance than any other element. The second type is the engineering student who during his freshman year needs an introduction to calculus and to vectors, as a preparation for physics and mechanics, rather than most of the work in college algebra. Partial fractions are best considered preliminary to certain integrations; the binomial theorem fits into the study of series; permutations and combinations are only of the most indirect value. The third type is the student who would later specialize in mathematics, and for him the present outline includes everything that he now gets in his freshman courses of trigonometry, algebra and analytics with the exception of permutations, a loss more than counterbalanced by his knowledge of calculus.

THE DUPLICATION PROBLEM.[1]

By JAMES H. WEAVER, West Chester High School.

There have come down to us from the remote past three problems of perennial interest, namely, the duplication problem, the trisection problem, and the quadrature problem. The first of these has for its object the finding of the edge of a cube that is double a given cube, the second the trisection of any angle, and the third the finding of a square equivalent to a given circle. It is the object of this paper to give:

I. A short historical sketch of the duplication problem, calling attention to the various methods of attacking the problem that were made possible by the advancement of mathematics.

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[1] The following authorities may be consulted for a fuller discussion of the problem:

(*a*) *Collections of Pappus*, ed. Hultsch, Berlin, 1876, page 31 and ff. In addition to the approximate solution mentioned in the text, this contains the principal solutions of the Greeks of the Alexandrian School.

(*b*) *Historia Problematis de Cubi Duplicatione*, by N. T. Reimer, Göttingen, 1798, 8vo, 238 pages. A very full account of the history of the problem.

(*c*) *Historia Problematis Cubi Duplicandi*, by C. H. Biering, Copenhagen, 1844, 4to, 64 pages. This was largely stolen from (*b*) according to S. Günther. See Cantor, *Geschichte der Mathematik*, Volume 4, pages 28–9.

(*d*) W. W. R. Ball, *Mathematical Recreations and Essays*, New York, Sixth Edition, 1914, pages 285–291. The discussion here is merely an outline of a few of the most noted solutions with references to the original sources.

(*e*) *Das Delische Problem*, A. Sturm, Linz, 1895–7. 4to, 140 pages. A full discussion of the problem in all its stages of development.

(*f*) Cantor, *Geschichte der Mathematik*, Vol. I, first edition, page 349. We have here an authoritative historical account of some of the solutions of the Greeks.

(*g*) Article by A. Conti, found in the following books: *Questioni Reguardanti le Matematiche Elementari*, Volume 2, Bologna, 1914, pages 185–231, and *Fragen der Elementar Geometrie*, Theil 2, Leipzig, 1907. Both these books were edited by Enriques. This article gives a very good discussion of some approximate solutions and takes up the question of the impossibility of a solution by means of ruler and compasses.

(*h*) *Famous Problems of Elementary Geometry*, F. Klein, translated by Beman and Smith, Boston, 1897. This volume includes a discussion of the impossibility of the problem by means of ruler and compasses. We will refer to the above mentioned accounts by the names of the authors; thus, Sturm, p. . . ., etc.